

The Hierarchy of Cosmic Voids

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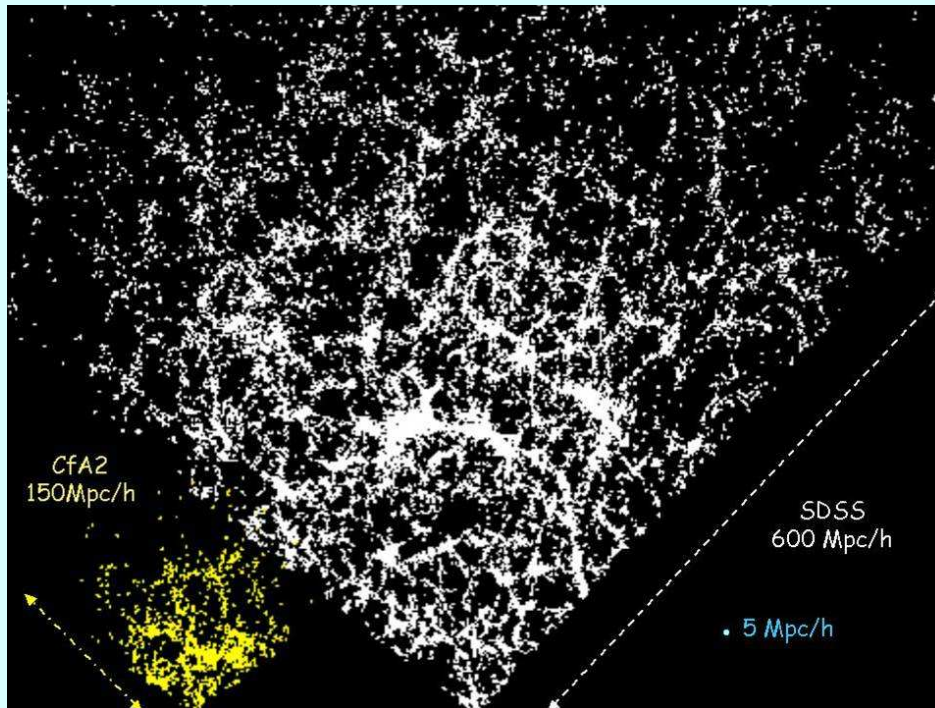
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PLAN OF THE TALK

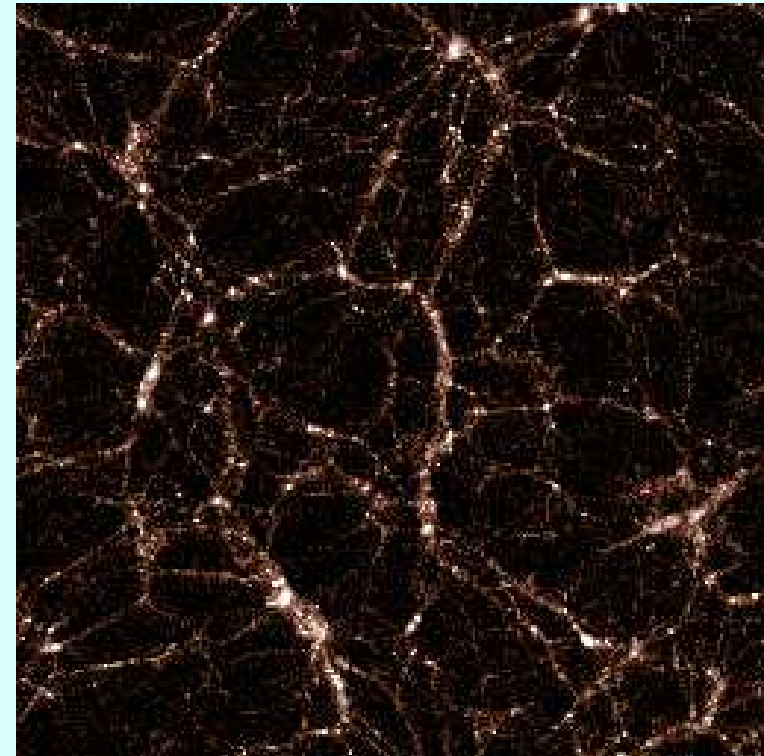
1. Voids in the Cosmic Structure
2. The hierarchy of fractal voids
3. Zipf's Law
4. "Cut-out sets"
5. Void finders
6. The Zipf law for Cosmic Voids
7. Voids in a multifractal
8. Conclusions

The Cosmic Structure

Matter clustered on large scales \rightarrow large voids (50 Mpc/ h in Boötes)



Structure displayed by old and new galaxy surveys



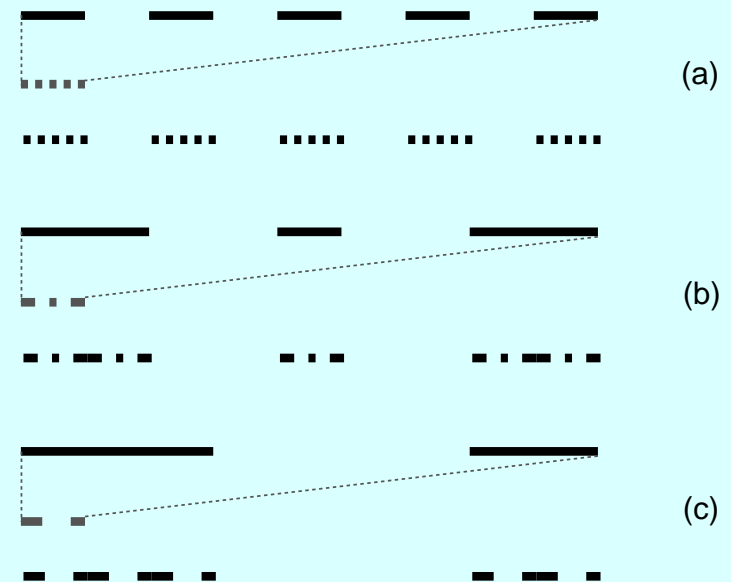
“Cosmic Web” produced by N-body cosmological simulation

Fractal voids

Mandelbrot introduced *tremas* = fractal holes

- Morphology related to void structure \rightarrow *lacunarity*
- Same dimension fractals can have different void structure

Simple in one dimension:



Three Cantor type fractals and their gaps

Zipf's law

- Mandelbrot: number of length $U > u$ voids fulfill

$$N(U > u) \propto u^{-D} \Leftrightarrow \text{Zipf's law}$$

- General Zipf's law: size $\propto \text{rank}^{-\alpha}$.

- $N(U > u)$ is the rank: $u \propto \text{rank}^{-1/D}$.

- Zipf's law is equivalent to the probability of finding a given size be a *power law*.

Cut-out sets

- Mandelbrot and Falconer's "cut-out sets": obtained from an open interval by removing an infinite sequence of disjoint open intervals.
- The *box-counting* dimension depends only on the void set.
- Every one-dimensional fractal is a cut-out set.
- **Theorem:** E , with a sequence of *ordered* void intervals ℓ_k ($k = 1, 2, \dots$), such that $\sum_{k=1}^{\infty} \ell_k = |E|$, has

$$\dim_B E = -1 / \lim_{k \rightarrow \infty} (\log \ell_k / \log k).$$

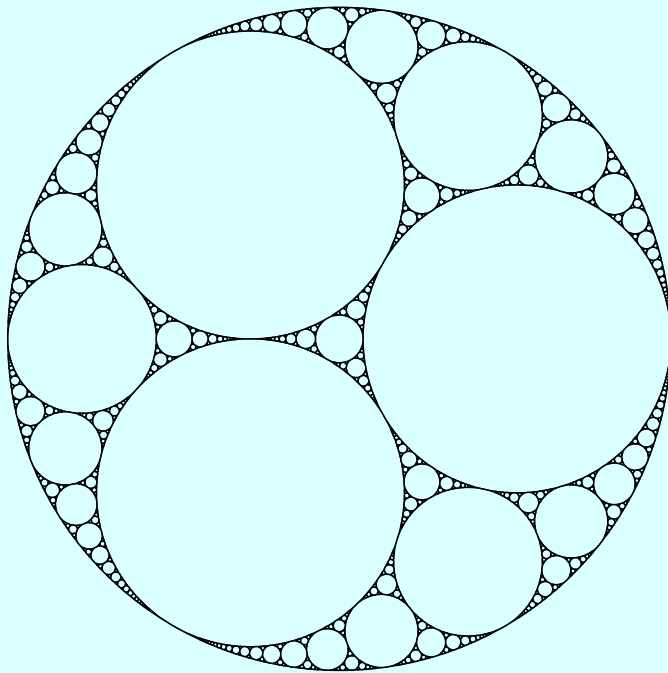
Equivalent to Zipf's law $\ell_k \sim k^{-1/D}$, $D = \dim_B E$.

Example: Cantor set

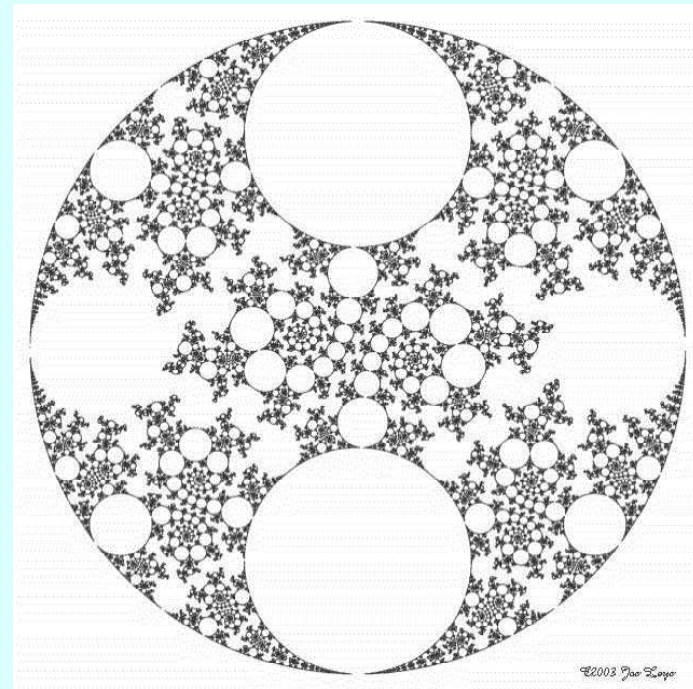
- Cantor set obtained by removing the sequence of (disjoint) open middle-third intervals.
- Length $\ell_k = (1/3)^n$, $2^{n-1} \leq k < 2^n \Rightarrow \frac{\log \ell_k}{\log k} = \frac{n \log(1/3)}{(n-1) \log 2} \Rightarrow \dim_B E = \frac{\log 2}{\log 3}$.
- In this case, $\dim_B E = \dim_H E$, but this does not hold in general. Take the set $E = \{1/3, 1/3+1/9, 1/3+1/9+1/9, 1/3+1/9+1/9+1/27, \dots\}$. It has $\dim_H E = 0$, since it is countable.
- In general, arrangement of voids matters.

Fractal voids in more than one dimension

- Voids in $d > 1$ can have different **shapes**; but $\dim_B E$ still depends only on the sequence of sizes.
- In $d > 1$, cut-out sets must have **topological codimension one**: curves in $d = 2$, surfaces in $d = 3$, etc.



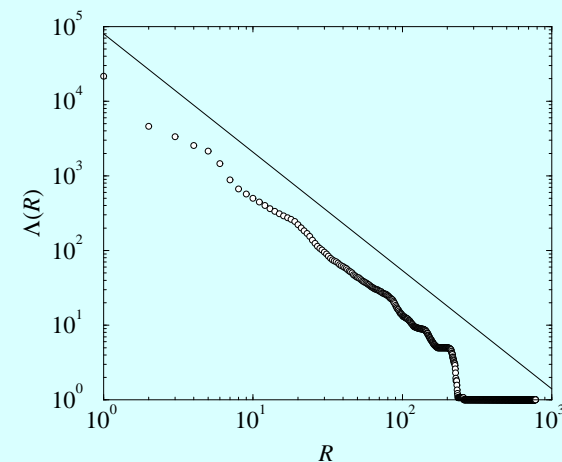
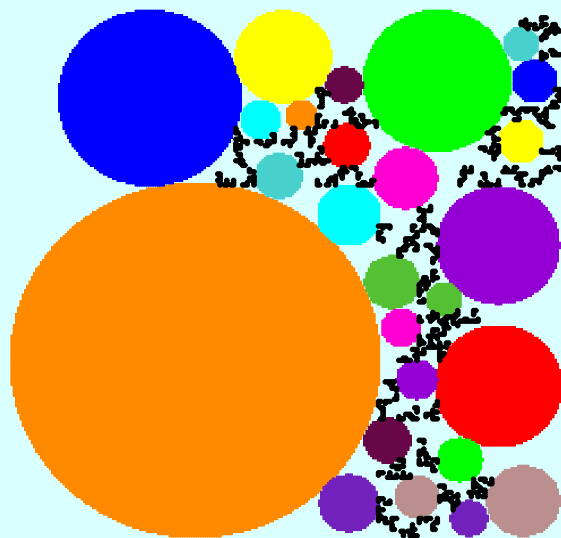
Apollonian packing of disks



Limit set of a Kleinian group

Voids in arbitrary fractals

- Gaité & Manrubia, 2002, MNRAS: Voids of constant shape: discs, squares, etc.



Random Cantor fractal, its voids, and Zipf's law

- **Drawback:** Constructed cut-out set can have larger dimension. With discs, we need $\dim_B \text{fractal} > \dim_B \text{cut-out set} (1.31)$.

Solutions:

- ◆ Voids must touch the fractal.
- ◆ Voids of arbitrary shape \rightarrow void finders.

General cut-out sets

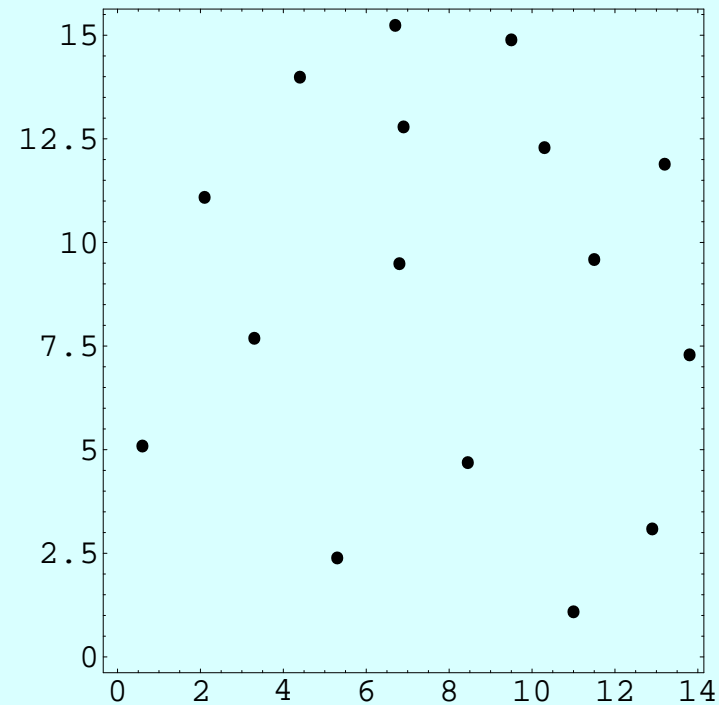
- Voids of arbitrary shape can have fractal boundary \Rightarrow too general.
- Simple restriction: **convex** voids (as in, e.g., the “Voronoi foam” of Icke & van de Weygaert, 1987).
- Theorem (Gaite, 2006, Physica D): Cut-out set $E \in \mathbb{R}^d$ with $d - 1 \leq \dim_B E < d$ and **convex non-degenerate** voids fulfill Zipf’s law with exponent $d / \dim_B E$.
Non-degenerate (in $d = 3$): shape-coefficient a_k^3 / v_k^2 is **bounded**.
- **Furthermore**: for a statistically self-similar cut-out set, $\dim_H E = \dim_B E$ almost surely.

Void finder based on discrete geometry

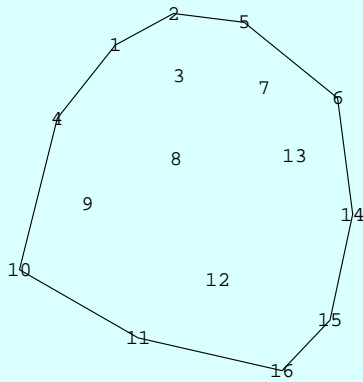
Computational problem: in a finite set of points, find filling voids of variable shape \rightarrow criterium for separation.

Methods of discrete stochastic geometry: **Delaunay and Voronoi tessellations**

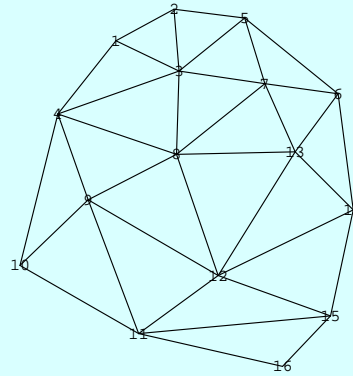
Consider a set of points
(in the plane):



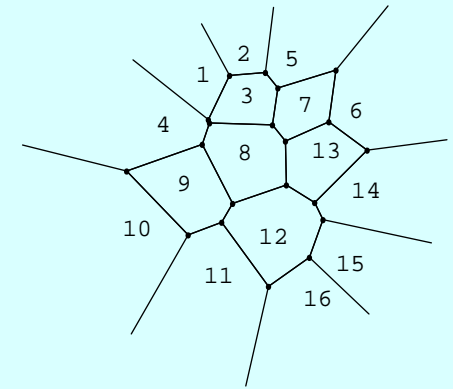
Void finder based on discrete geometry



Convex hull



Delaunay
triangulation



Voronoi tessellation

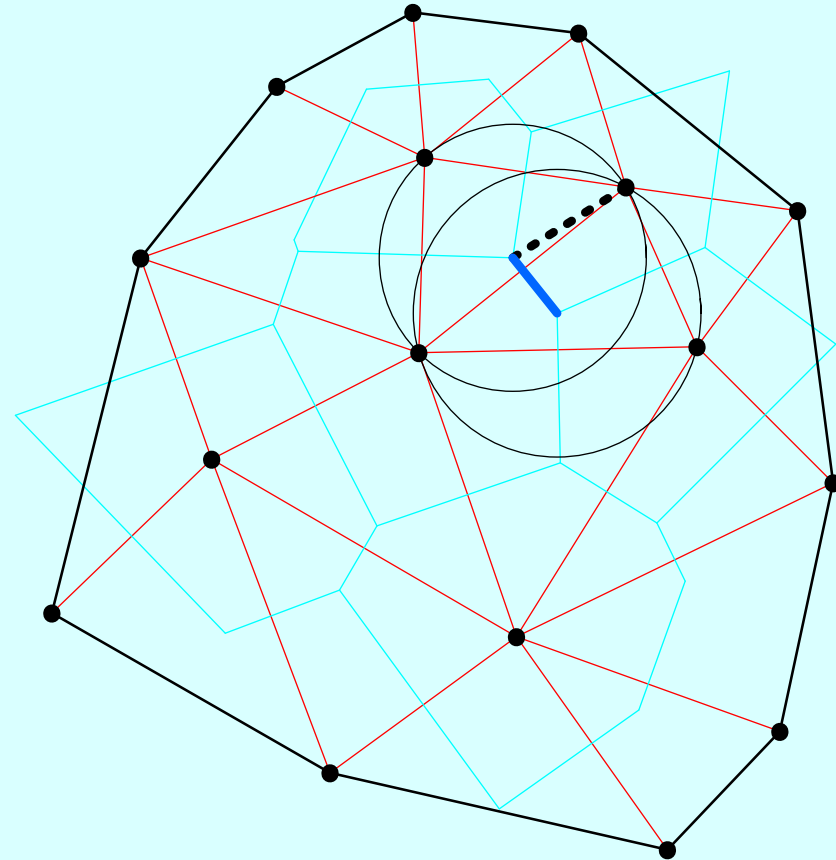
- The circle circumscribed to each Delaunay triangle does not contain more points of the set.
- The Delaunay triangulation maximizes the minimum angles.

Delaunay triangulation \leftrightarrow Voronoi tessellation.

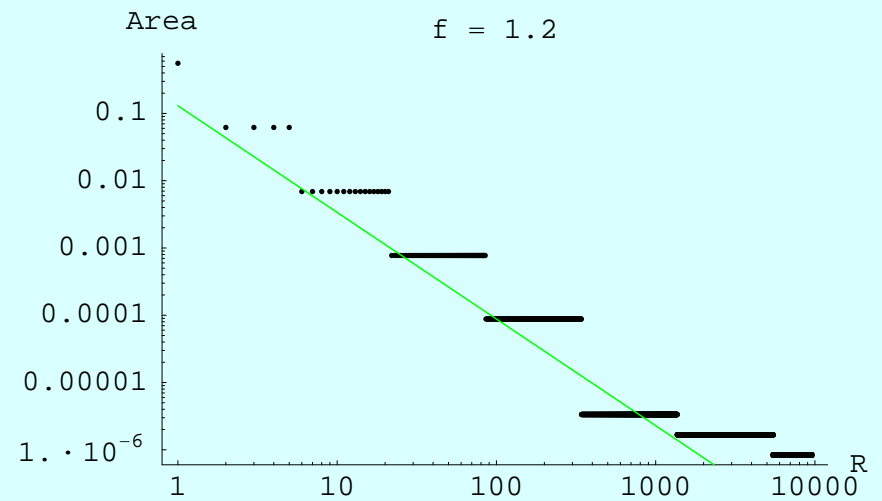
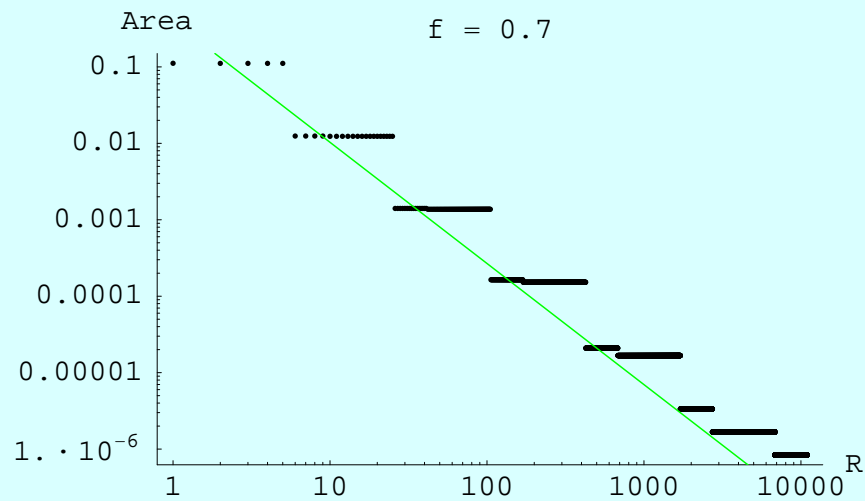
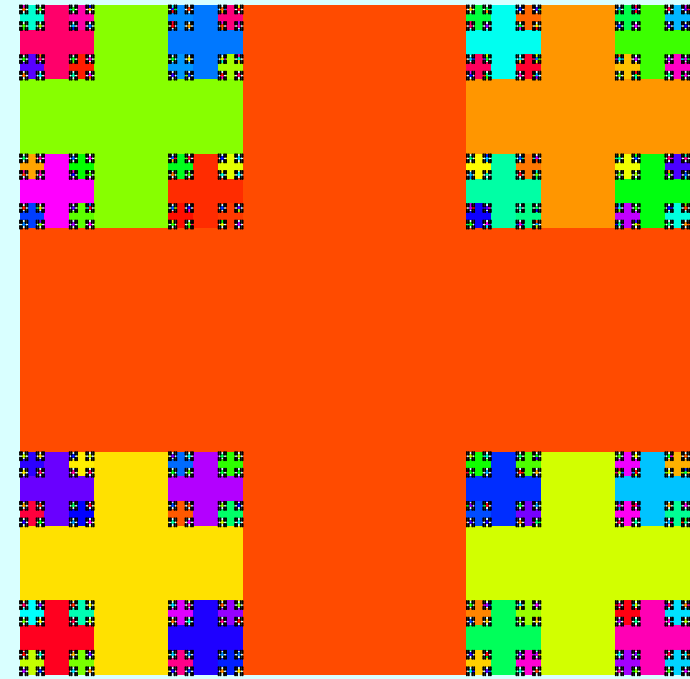
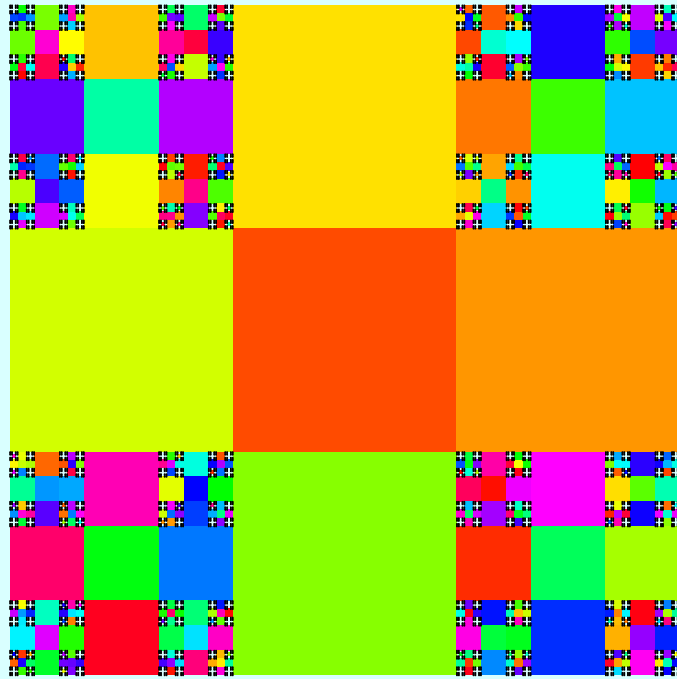
Void finder method

One criterium to join *basic voids* (Gaite, 2005, EPJB):

Circles circumscribed to two adjacent triangles, with the segment linking their centers and the radius of the smaller circle \rightarrow **overlap parameter** f is a bound to the ratio of both lengths.

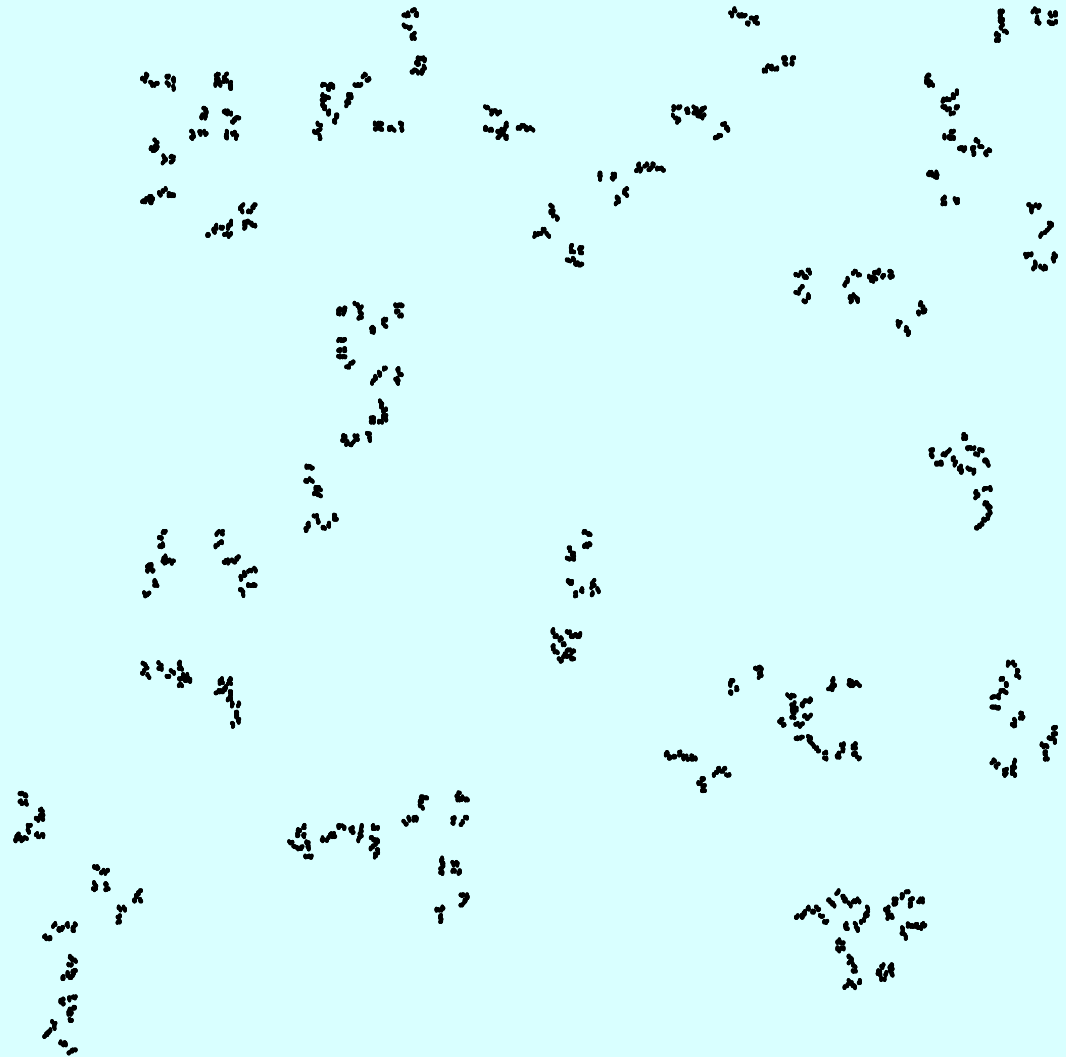


Application to self-similar fractal



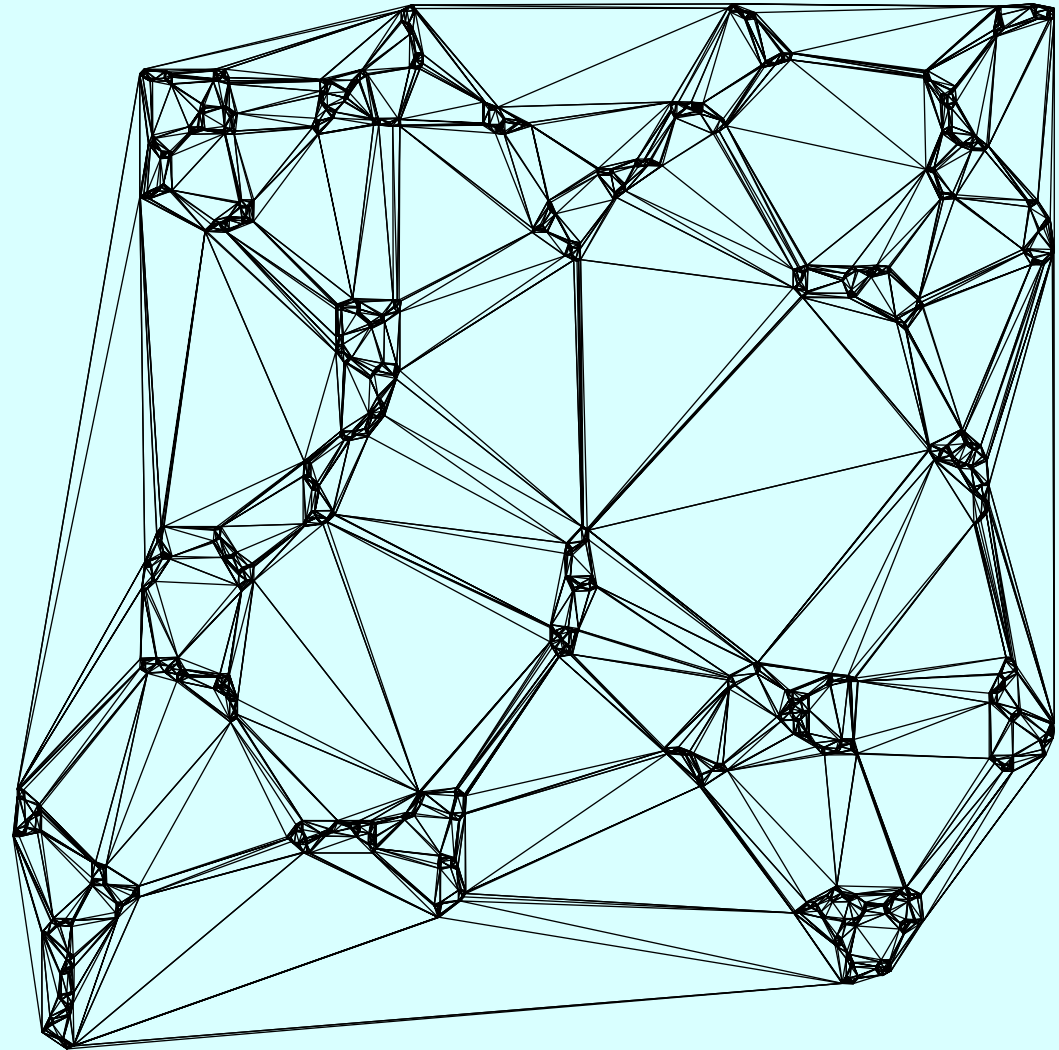
Random fractal: Example 1

Random Cantor fractal
(12 288 points) with
 $D = 1$ and transition to
homogeneity.



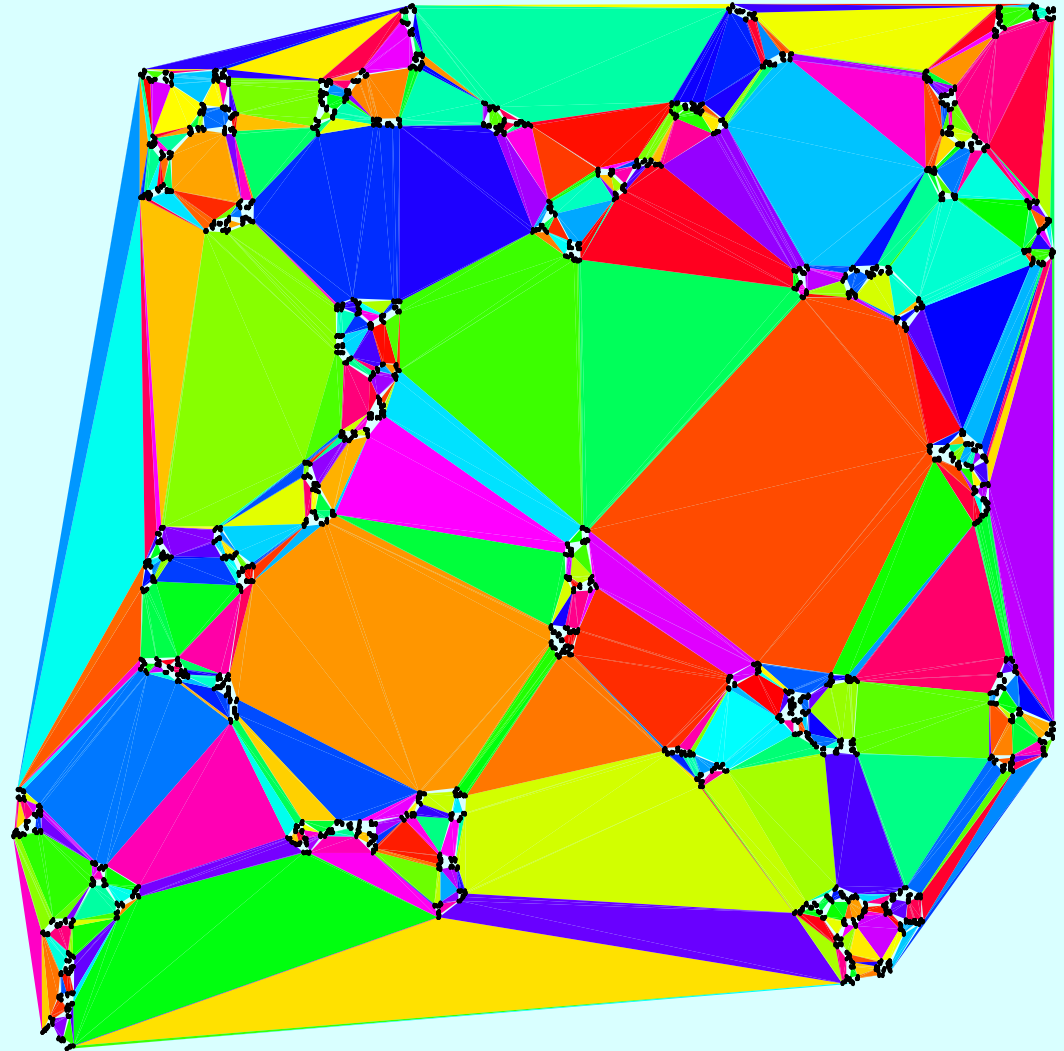
Random fractal: Example 1

Delaunay triangulation
of the fractal set:

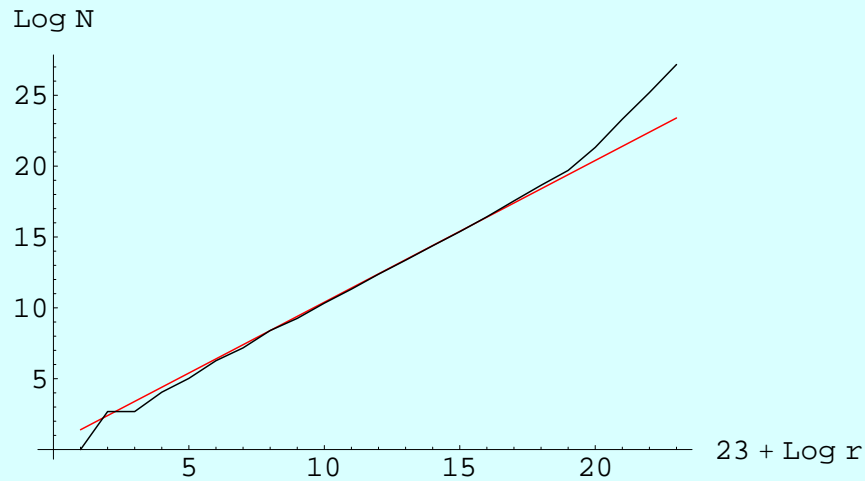


Random fractal: Example 1

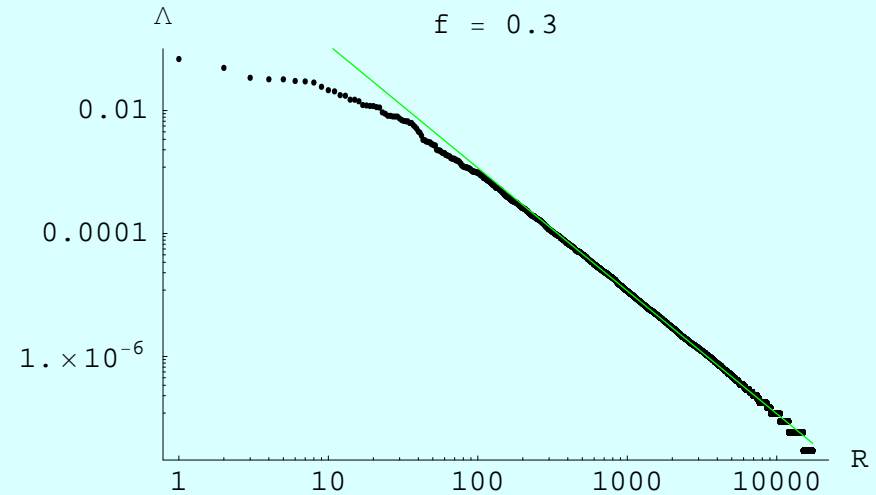
Largest 500 voids with
 $f = 0.3$ overlap:



Random fractal: Example 1



$N(r)$ and fit $D = 1$



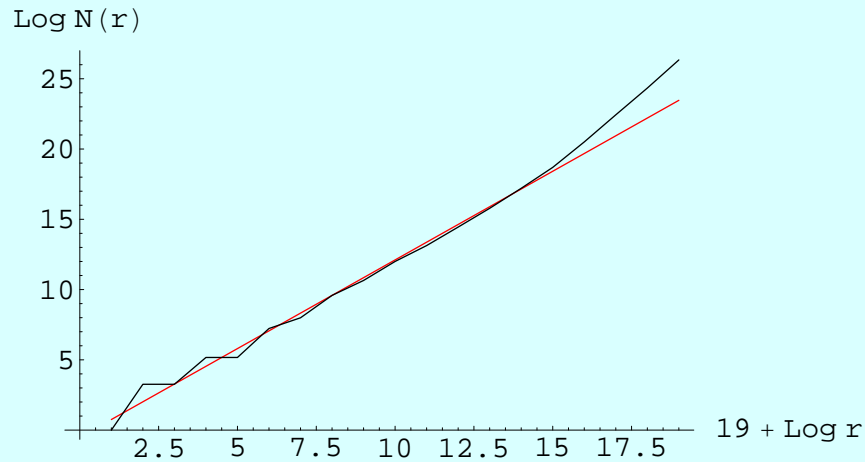
Zipf's law with slope $-2/D$

Compare:

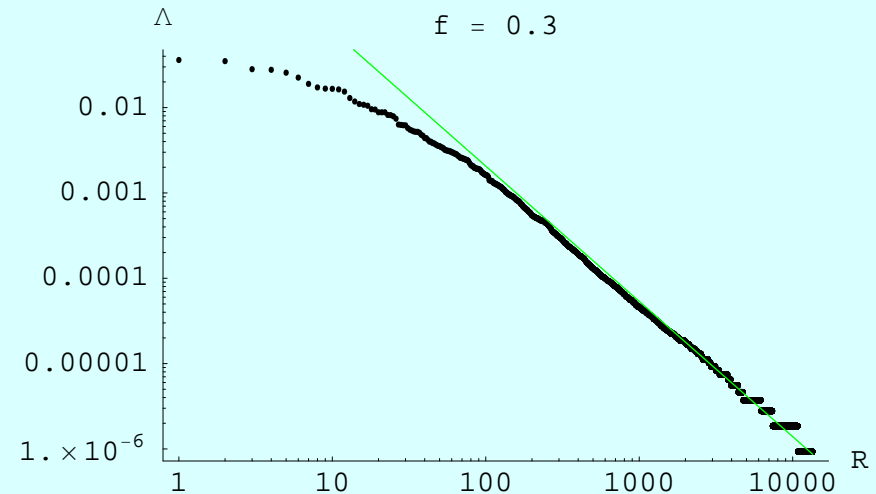
- Scaling ranges of $N(r)$ and $\Lambda(R)$.
- Transition to homogeneity.

Random fractal: Example 2

Random fractal (9 216 points) with $D = 1.26$



$N(r)$ and fit $D = 1.26$



Zipf's law with slope $-2/1.26$

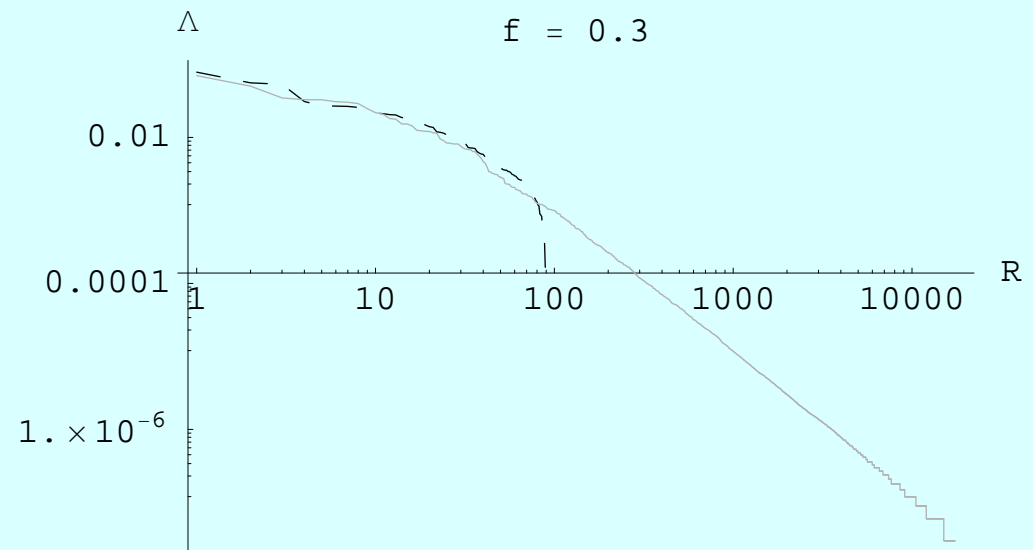
- Topological dimension 0 \Rightarrow voids ill-defined.
- Cut-out set = Fractal + boundary of voids.
- Minimum value of D is 1 (boundary of voids).

Transition to homogeneity

If the fractal becomes homogeneous ($D = 3$) on large scales \Rightarrow
flattening of Zipf's law in the rank-ordering of voids.

Transition to homogeneity in Example 1:

Flattening of the Zipf law and voids of a random distribution of points (dashed line) having small-rank voids of the same size ($f = 0.3$ for both).



Fractal measures

Basic measure is the number-radius relation $N(r) = B r^D$,
 $r < r_0$ (scale of transition to homogeneity $D \rightarrow 3$).

■ Gamma function (two-point correlation):

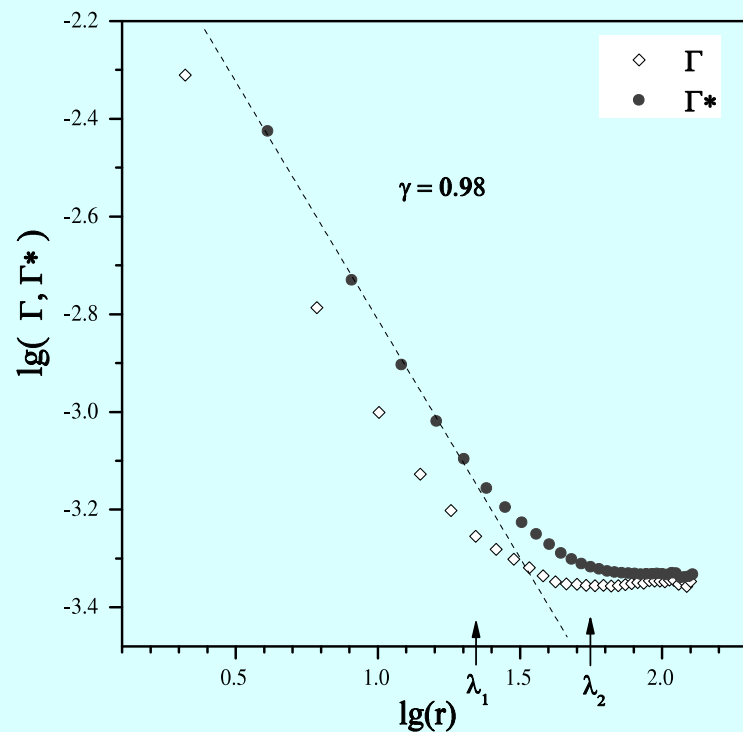
$$\Gamma(r) = \frac{1}{4\pi r^2} \frac{dN(r)}{dr} = \frac{BD}{4\pi} r^{D-3}.$$

One defines $\gamma = 3 - D > 0$.

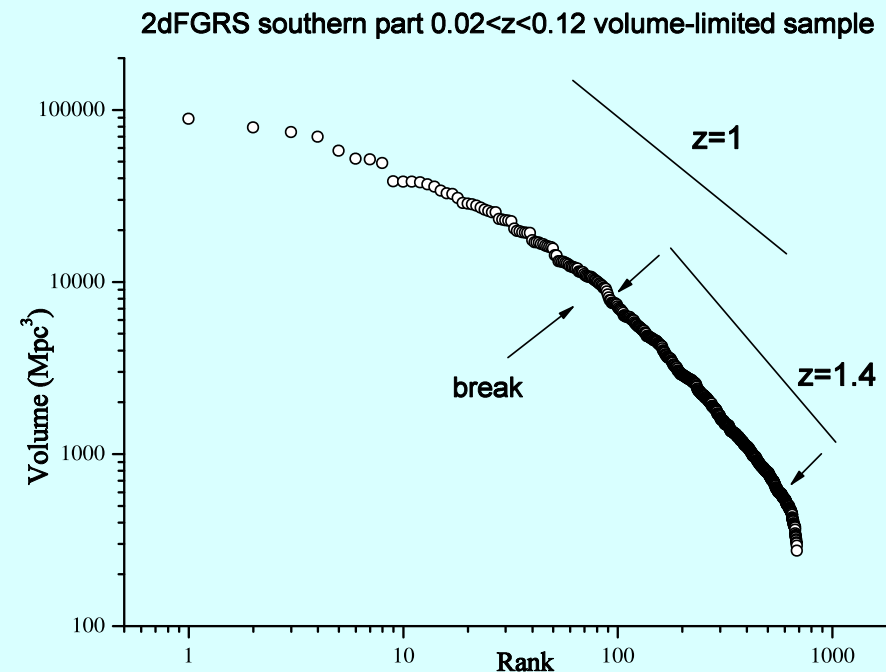
■ Gamma-star function:

$$\Gamma^*(r) = \frac{N(r)}{4\pi r^3/3} = \frac{3B}{4\pi} r^{D-3}.$$

Scaling of galaxy clusters and voids



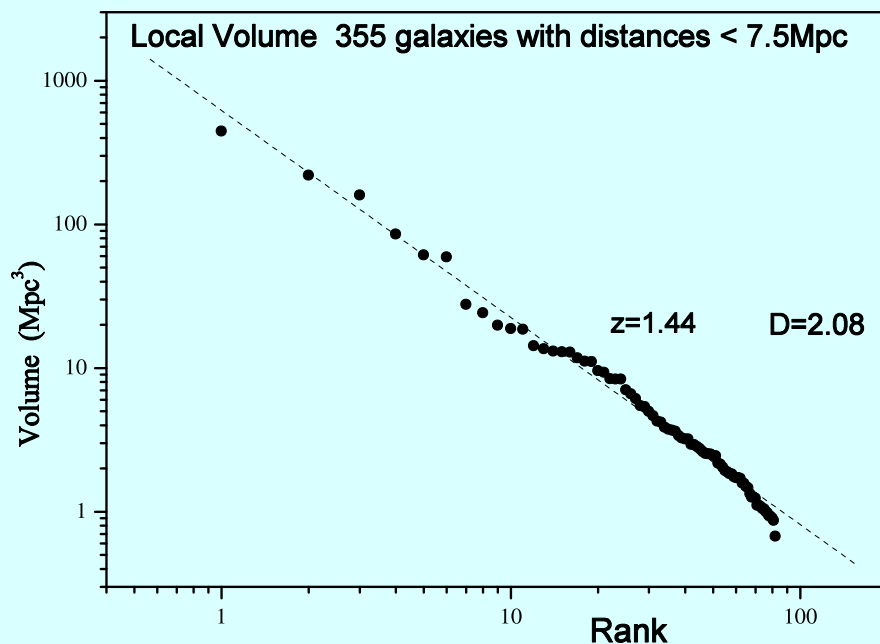
$\Gamma(r)$ of SDSS VL sample.
 $r_0 \simeq 15$ Mpc (Tikhonov).
Slope $\gamma \rightarrow D \simeq 2$



The Zipf law for a 2dF VL sample (Tikhonov). Slope $z \Rightarrow D \simeq 2$

Scaling of galaxy clusters and voids

Rank ordering of local “minivoids” (Tikhonov & Karechentsev, 2006, ApJ):



Slope $\gamma \Rightarrow D \simeq 2$. No transition to homogeneity

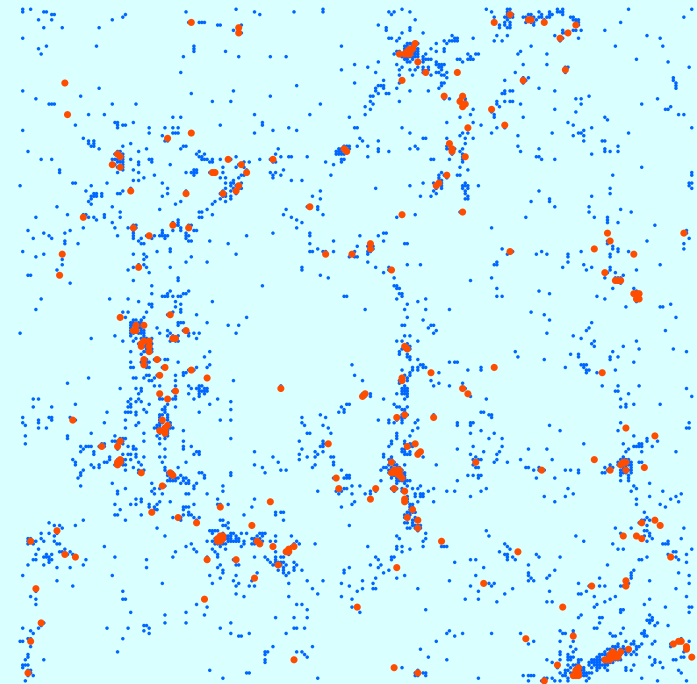


Six largest minivoids within the sphere of radius 7.5 Mpc

Voids in a multifractal

Full dark-matter distribution is multifractal (Gaite, 2007, ApJ) \Rightarrow various (halo) populations with different clustering and fractal dimension.

Two halos populations in the GIF2 simulation (massive in red and light in blue). More clustered massive halos leave larger voids.



- **Faint galaxies** in voids \leftrightarrow galaxy formation (Peebles, 2001).
- Distribution of dark matter inside voids (Gottlöber et al, 2003).

SUMMARY and CONCLUSIONS

- Clustering \Rightarrow hierarchy of voids. Scaling voids \leftrightarrow Zipf's law.
- Rigorous definition of fractal voids \rightarrow cut-out sets, with topological codimension 1.
- Zipf's law flattening for largest voids \rightarrow transition to homogeneity.
- Void finders \rightarrow voids in fractals with topological codimension ≥ 1 (addition of boundaries).
- Recent analysis: large voids which fulfill Zipf's law but yield minimum $D = 2$.